Classes of Strategy-proof Mechanisms for Preference Revelation for or against a project

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Abstract: Facing public projects with consequences of difficult to assess with markets, as for instance in environmental conflicts the stakeholders react according to two types of values: (i) their direct vested interests that can be in certain case compensated in economic transactions and (ii) their moral attitudes which is not prone to any monetary valuation. In the later case, the usual way to get preferences expression in society is voting which ignores the strength of preferences. The economic and political literature has shown since long that the range of strategy-proof, non-manipulable, mechanisms, as well in voting as in mechanisms allowing for transfers is very slim[sg1]. However in the special case of the decision over a public project, we propose here a set of mechanisms and we show that there are no other strategy-proof mechanisms out of this class. This class is much wider than the classical Groves-mechanisms and, contrarily to Green & Laffont who have precluded feasible solutions with non-linear utility and pareto optimality. We establish a general necessary condition for strategy-proofness requiring a different definition of Pareto-optimality. The mechanisms are based on the properties of commutative groups in algebra and can be simply implemented. In particular a pure voting procedure (i.e. a quasi-referendum) belongs to this class.

The interest of this approach is to design systems of penalties in pivot-mechanisms in which the role of money is mitigated and the focus is made on the non-monetary change in welfare, suffered or benefited by the stakeholders. We think that this can open a way to formalize systems of compensations in environmental issues, used in some countries to exhibit and internalize the liability of pivot stakeholders in decisions, without jeopardizing preference revelation.

Keywords: Groves mechanisms, non-market valuation, environmental preferences, commutative groups, voting theory, referendum, non-linear utility

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1. Motivation

The increasing occurrence of tough local environmental conflicts about big public or private projects challenges policy decision-making when the range of individual valuations includes extreme positions. The conflict about the airport project of Notre-Dame-des-Landes in France is an example where individual and institutional attitudes were strongly opposed, and where after a series of unsuccessfully attempts, spreading over decades, to make a decision based on various types of consultation (including a referendum) at regional and national levels, the upshot has been a "dictatorial" decision by the President himself. Similarly, in two conflicts in the Mediterranean region concerning (i) the rejection of industrial wastes in the *Parc National des Calanques* near Marseille, and (ii) the construction of a large power plant burning biomass, we have recorded that the distributions of expressed positions of stakeholders, far from a Gaussian shape, are both U-curved. Here also, the disputes in spite of several local as well a national decisions are still unsettled, with cases pending in courts. They have been also widely echoed in the national opinion, and the Government itself has been jolted by strong internal dissensions about the case. Moreover, it has been found that the opposition to the two projects were stronger among people far from the location of the project, where the environmental consequences would not be not suffered most, showing that the values not always linked to direct economic vested interest (Boutin *et al.* 2019).

This impotence to settle such conflicts holds to the fact that environmental issues are the realm of externalities of different sorts, and their welfare effects on individuals of the present and future generations, are multiform and complex and therefore difficult to aggregate for public decisions. In the three disputes mentioned above, one of the major stakes is the harm made to biodiversity by the implementation of a project. Biodiversity is a complex and intricate object interpreted differently by scientists, industrialists, civil servants, or religious, and which involves people's values towards nature.

Actually, the welfare effects in those conflicts are of both types: on the one hand, direct tangible welfare consequences that can be assessed to some extent with the tools provided by economic approaches; economic non-market valuation methods are welcome in this respect. On the other hand, preferences are founded on intangible elements sustained by moral values. For instance, certain defenders of biodiversity are driven by their vision of the non-human living beings: they would resent as an intolerable detriment to *their* welfare, the knowledge they have of the treatment of calves in slaughterhouses or of living in a country where corridas are allowed. It is difficult to assume that monetary compensation could soothe their anger.

This debate is not specific to environmental disputes: societal issues such as that death penalty, abortion, or civil rights show similar dilemmas. In these cases, unthinkable monetary compensation of the losers has to leave place to political debate before some kind of referendum or Parliament decisions. In short, economic methods do recognize the intensity of preferences while political ones do not.

Compensations or monetary transfers are subject to objection on the basis of a general principle of justice reminded by Rawls (1971, p 49):

> *Each person possesses an inviolability founded on justice that even the welfare of society as a whole cannot override. For this reason justice denies that the loss of freedom for some is made right by a greater good shared by others. It does not allow that the sacrifices imposed on a few are outweighed by the larger sum of advantages enjoyed by many. Therefore in a just society the liberties of equal citizenship are taken as settled; the rights secured by justice are not subject to political bargaining or to the calculus of social interests. …an injustice is tolerable only when it is necessary to avoid an even greater injustice…justice (is) uncompromising.*

In environmental disputes, aside from vested interests of some, many stakeholders invoke a kind of "personal inviolability" of beliefs and values, theirs or those of others, and they would not see the salute in compensation and economic approaches. Their expectations are more satisfied by voting procedures.

The usual expression of justice in political decision is voting (either directly or indirectly via representatives). Voting can be viewed as the extreme non-market valuation method, which discards the intensity of individual preferences and we shall show how to formalize this idea. Voting methods are all based on some distribution of voting rights among the stakeholders of the issue. Of course they may involve inequality of representation but far from being a measure of preferences strength, this inequality results from political decision (in the USA, the citizens are unequally represented in the election of the President of the nation, and this is a consequence of how the constitution has been designed, not a reflect of strength of preferences.) Actually In modern political systems, the principle "one-man-one vote" among the stakeholders of a case, is broken in many ways (age or citizenship restriction for instance, multilevel representation by agents, delegates, representatives), which reflect the desire to leave the decision to people who have better knowledge and information from where stems a greater wisdom in the way to aggregate the preferences of the principals that have mandated them for honest representation. For instance the reluctance in France in 2019 to increase the role of referenda in public decisions, in spite of a strong societal demand in street demonstrations reflects the warnings by many political scientists of the perverse effects of the one-man-one-vote-principle. Whatever the method, referendum or multilevel-representation, both ignore the assessment of the strength of preferences. If preferences over the set of alternatives states s are captured by a utility function $U(s)$, this should be unique up to a monotonic positive transformation.

In the academic literature, economic and political approaches of citizens' preferences have been mostly treated separately. Although several generations of economists since Arrow'essay (1952) have contributed to political voting theory, there remains a distance between those who deal with the *intensity* of preferences for commons, public goods, or state of the environment, and prefer to keep the market language even if the market fails, like in public goods or commons, and those who are only interested in *power distribution*, and consider voting systems with no reference to the (market or non-market) monetary value of the social outcome.

This discrepancy holds to the nature of the mathematics required for modelling in both cases! The economists' privileged kit tool is *mathematical analysis*, because they work in real time-spaces ($\boldsymbol{T}\mathrm{x}\boldsymbol{R}^{\boldsymbol{n}}$) where they can "optimize", using convex analysis, differential and integral calculus, etc. By contrast, voting theories favor *combinatory algebra*. In other words, the economists do like *continuity* while the voting theorists favor *discrete* structures (binary relations, structure of coalitions etc.). The two approaches have therefore to mobilize two distinct branches of mathematics, often difficult to reconcile. However, algebra is more general than its combinatory restriction and can be mobilized for building a bridge between the two approaches. We propose here a link between the two fields by focusing on the direct collection of preferences in public decisions satisfying a specific condition: they should be free of any incentive to cheat or distort them for strategic or tactical reasons. The theorists of voting call them "strategy-proof" while the economists prefer to say "incentive-compatible". Being rather game-theoretic oriented, we prefer using the first term.

2- Strategy-proofness the formal setting in the literature

Using collective choices procedure that are not strategy-proof raises complex societal questions: Since people have to anticipate or figure out others' choice before making their own, they engage into information searching whose efficiency depends their available resources for efficient search and their state of knowledge about the society . Then the collective choice becomes determined more by sociological factors than by the pure mechanics of the procedure. In the disputes mentioned above, clear strategic postures were adopted by certain stakeholders, who have tended to exagerate knowingly their declared position to push up their favored solution in the game. So power in the society does matter: a handful of threatening tiny group may upset a collective decision. This is why there are still reasons for a more comprehensive investigation of strategy-proofness, if only for collecting an unbiased view of the preferences of the stakeholders.

In the literature, a procedure for social choice consists of a space of admissible messages, within which each stakeholder picks one and sends it to the central planner according to the state of his preference for a social outcome. The social outcome may or not involve compensations or contributions. The classical "revelation principle" can be invoked here to restrict the message space $\boldsymbol{U}$ for each agent to that of all admissible utility functions. A joint-message collected by the central planner is denoted by $\boldsymbol{u} = (u_1, \dots, u_i, \dots, u_n)$, and is called a *profile* (of utility function). $\boldsymbol{U^n}$ is the Cartesian product of n individual message spaces $\boldsymbol{U}$. The central planer computes the social outcome with two functions:

$\delta(\boldsymbol{u}) : \boldsymbol{U^n} \rightarrow \; Y$ determining the social-outcome

$\boldsymbol{\tau}(\boldsymbol{u}) : \boldsymbol{U^n} \rightarrow \boldsymbol{R^n}$ defining a vector of transfers of some private good to the stakeholders. Stakeholder i is assigned the component $\tau_i(\boldsymbol{u})$ of $\boldsymbol{\tau}(\boldsymbol{u})$.

This formulation can capture both procedures with or without transfers like voting in which the second function is empty.

For a given profile $\widehat{\boldsymbol{u}}$, the procedure generates an n-person-game, where $\boldsymbol{U}$ is the set of individual messages available to each player and the utility attached by player i to any joint message $\boldsymbol{u}$ in $\boldsymbol{U^n}$ is given by: $\widehat{U}_i(\delta(\boldsymbol{u}), \tau_i(\boldsymbol{u}))$

This n-person game may or not have equilibria of different sorts Nash, Stakelberg, randomized strategies etc.). The literature on strategy-proofness focuses on procedures such that $\widehat{\boldsymbol{u}}$ is a unique dominant strategy equilibrium of this game. In such games, whatever his own utility u_i stakeholder i has never to anticipate or know the others' message to make his choice and therefore has no reason to choose $\hat{u}$, when it is his sincere preference.

Formally, remind that a strategy $u_i \in \boldsymbol{U}$ is dominant (DS in short) for player i if:

$$u_i\,[\delta(\,u_i, \boldsymbol{u}_{-i}), \tau_i(\,u_i, \boldsymbol{u}_{-i})] \;\geq u_i\,[\delta(u_i', \boldsymbol{u}_{-i}), \tau_i(\,u_i', \boldsymbol{u}_{-i})],\; \forall u_i' \in \boldsymbol{U}\; and\; \forall \boldsymbol{u}_{-i} \in \boldsymbol{U^{n-1}}$$

A dominant-strategy equilibrium (DS equilibrium in short) is one in which each player has selected a dominant strategy. So we look for procedures such that any sincere valuation $u_i\,(.)$ is a dominant strategy whatever the profile **u**

However a DS-equilibrium does not guarantee Pareto-optimality of the outcome (a classical example is prisoner's dilemma). So, an additional condition for society is that, at a DS equilibrium, whatever the profile of preferences, one cannot have

$u_i\,[\delta(\,u_i'\,,\boldsymbol{u}_{-i}),\tau_i(\,u_i'\,,\boldsymbol{u}_{-i})] \;>\; u_i\,[\delta(\,u_i,\boldsymbol{u}_{-i}),\tau_i(\,u_i,\boldsymbol{u}_{-i})]$ for some i and some u_i'

3 -Main classical results

In the political approach, usually referred to as social choice theory, admissible preferences $U(y)$ were considered by Arrow (1952) as any complete preorders over a set A of alternatives. Transfers were precluded. In this case the utility function $U(y)$ just reflects a complete preorder on A and is unique up to an arbitrary positive monotonic transformation. In this case the search for a procedure subject to some reasonable conditions and providing an aggregated complete preorder of the alternatives was shown by Arrow as vain as soon as there are more than two alternatives, except if there is a "dictator". Condorcet (1793), discussing the quest for new democratic procedures in the emerging constitution after the French revolution, had already exposed this impossibility, although without formally proving it, but by using the illustration of the well-known "paradox" named after him. Finding a consistent procedures required to restrict the admissible preferences, for instance to the so-called "Blackian orders".

However, Arrow was not preoccupied with strategy-proofness (he explicitly tells so in his original essay). Later, the Gibbard-Satterwaite theorem (1972) came to establish the impossibility of the existence of non-dictatorial strategy-proof voting procedures when the social choice is made by voting among more than three alternatives. This proximity of both theorems was in fact observed very early in the literature, and the reason is that they were founded upon the same algebraic structures (see Batteau-Blin-Monjardet, 1981): the core of the same induced game in both cases is empty, and therefore there is no DSE equilibrium revealing truth preferences (excluding dictatorial procedures).

In the economic approach, assuming that the states of the world include both public and private goods, the scope of admissible preferences can be restricted by standard assumptions of neoclassical models. In spite of this reduction, the hopes for strategy-proof collective choice procedures were severely curbed by Green and Laffont (1977) who showed that except when the utility function takes a special linear form $U_i(y, t_i) = V_i(y) + t_i$, (linear separability in the public and private good) there are no efficient strategy-proof procedure that can fully reveal $V_i(y)$. They showed also that when the condition of separability is satisfied, any feasible strategy-proof Pareto-efficient mechanism necessarily belongs to a special class, named the G-mechanisms (after Groves ()), also introduced in Vickey () and Clarke ()), and therefore sometimes referred to as the GCV-Mechanisms. The Pareto-condition requires that the sum of the $V_i(y)'s$ be maximized.

A major consequence of such mechanisms has to be emphasized: although the linear utility function is exempt of income effect (preferences for the project do not depend on the amount of private good held), this maximization confers to each stakeholder a weight in the decision equal to the damage or benefit to his wealth expressed in *numeraire* (or private good). As mentioned earlier, this type of effect may not be desirable in the assessment of environmental consequences. So the alternative for collective choice seems to be only a voting system (referendum type), with utilities $U(.)$ just reflecting complete pre-orders (being unique up to any monotonic transformation). In this case there is no hope for collecting the intensity of preferences.

To remind it, in a Groves mechanism, with a linear utility function, each stakeholder sends her valuation $V(y)$ of the public good. The resulting public good and transfers are determined by:

$$y^* = \text{argmax}_{\text{y}} \sum_{i=1,n} V_i(y) \qquad (1)$$

$$t_i(V(.)= \sum_{\forall j \neq i} V_j(y^*) - Max_{/y}(\sum_{\forall j \neq i} V_j(y)\} + h_i(V_{-i}(.)) \qquad (2)$$

Where $h_i(V_{-i}(.))$ is any arbitrary function of the valuation functions $V(y)$ of others player

Equation (1) identifies the Pareto-outcome. In equation (2) the first term is the valuation of this optimal outcome by all agents but i; the second term computes what would be the valuation of an optimal provision of public good y determined in the absence of i and this difference measures the change in utility suffered by the society caused by the presence of agent *i.* The third term cannot affect the message sent by i since it is independent of i's utility.

In words, each agent has to pay what the change in welfare she inflicts to the society by her presence. If the optimal y is the same both in the first and second term, the agent is determining (or "pivot") and there is no change caused by his presence. She therefore suffers or receives no transfer other than the independent third component of (2).

The third term of equation (2) provides a way to scale the whole net proceeds. However, if the project has a cost that must be funded by this flow of revenues, the balance of the budget is not guaranteed. This is one of the major flaws of G-mechanisms.

Groves' theorem was that the sincere valuation function is *the unique dominant strategy* of the game and the procedure is strategy-proof; The G&L's result was that any strategy-proof efficient (Pareto) mechanism with a linear utility function belongs the Groves class of mechanism. Moreover, without a linear utility function and under the Pareto condition, there is no feasible procedure in any case, and in particular in the binary choice situation we are interested in.

However, G&L rejected the idea that revelation of sincere preferences with transfers made sense for non-linear utility[1] because of the "income effect". So the way they express the Pareto condition is therefore restricted to the linear case and the efficient outcome is the y^* maximizing the sum of valuation. In spite of this "efficiency", as it is well known, Groves' mechanisms do not guarantee a balanced budget if one wants to fund the production of the social outcome: there could be deficit or surplus, which cannot in any way be redistributed among the stakeholders without destroying the incentive property of the procedure. So, in this stream of literature, this appears as a *restricted* vision of Pareto optimality limited to the maximization of public good valuation with linear utility, but disregarding the possible shortage or excess of private good, which are in some sort wasted (Batteau, 1982).

4- Strategy-proofness conditions with general utility functions and two alternatives

In this paper, we focus only on binary choice. Extensions will be discussed shortly in conclusions. With two alternatives, we know that a mere voting process reveals true preference but not their intensity, while a Groves-type mechanism reveals the intensity but confers to each stakeholder a weight in the decision in relationship with its intensity since provided the expressed preference be strong enough the outcome will be inverted. Is there room in between for other type of strategy proof procedure? The objective is here to state the necessary conditions for strategy-proofness without any assumption on the form of the utility function. Examples of feasible procedures will be given and their implementation discussed.

The community of n stakeholders is facing the decision y to accept ($y = 1$) or reject ($y = 0$) a project. If there is no system of compensation or contribution, the decision can be made either by a "dictator" or a vote with a distribution of voting rights. Alternatively, one may be willing to assess the change in welfare that could result from the acceptance of the project. The pro-project stakeholders would feel an improvement and the anti-project ones a detriment to their welfare. Let $U(0,\ w_i)$[2] be the initial utility of stakeholder i where w_i is private holdings of a certain individual good which can be assessed in units in whatever instrument one wants: divisible goods, numéraire, bitcoins, Brownie points, medals, marks, bonuses or maluses. We just call this good "individual welfare unit" (IWU). At first we leave aside the issue of tradability of those instruments, discussed further.

[1] P429 : "*Clearly, such a mechanism can be used only if agents have separable utility functions"*

[2] Note that there is a rationale in this setting: the initial position is *status-quo* and the decision is to *adopt the project or not*. If the initial position were $y = 1$ and the decision to move to $y = 0$, the solution $(0, x_i^{*\prime}) = U_i(1,0)$ would be different. This is the result of the "income effect", which does not exist in a linear utility function. However, as long as one does not consider other transfers for financing the project, it suffice to see 0 as the project and 1 as the status quo.

The utility function is supposed to satisfy a standard condition of non-decreasing welfare with the private holdings:

$U_i(y, w_i^{"}) \geq U_i(y, w_i')$ iff $x_i^{"} > w_i' \quad \forall\, y, w_i', w_i^{"}$

After the decision, the utility of agent i has been changed in (U(0, w_i- τ_i^0) if the decision is y=0 or in (U(1, w_i- τ_i^1) if the decision is 1, where τ_i^0 *and* τ_i^1 are amounts of IWU withdrawn or added to the stakeholder's account and defined by the choice procedure.

To guarantee sincere declaration of the preferred outcome by a stakeholder, one must have as a first condition that the effect of the transfer should leave her between the extreme utilities achieved with no transfers, that is to say:

For a stakeholder whose preference is for 1 (pro-project), and depending on the outcome y

$y\ = 1 \Rightarrow\ U(1,\ w_i) \geq U(1, w_i - \tau_i^1(\ w_i)) > \mathrm{U}(0,\ w_i)$

which implies $\quad 0\ \leq \tau_i^1(\ w_i) < \mathrm{U}(1,\ w_i)$- $\mathrm{U}(0,\ w_i)$ (1)

$y\ = 0 \Rightarrow\ U(1,\ w_i) > U(0, w_i - \tau_i^0(\ w_i)) > \mathrm{U}(0,\ w_i)$

which implies $\quad 0 \leq \tau_i^1(\ w_i) < \mathrm{U}(1,\ w_i)$-$\mathrm{U}(0,\ w_i)$ (2)

and for a stakeholder preferring 0 (anti-project)

$y\ = 1 \Rightarrow\ U(0,\ w_i) > U(1, w_i - \tau_i^1(\ w_i)) > \mathrm{U}(1, w_i)$

which implies $\ 0 \leq\ \tau_i^1(\ w_i) < \mathrm{U}(0,\ w_i)$-$\mathrm{U}(1,\ w_i)$ (3)

$y\ = 0 \Rightarrow\ U(0,\ w_i) > U(0, w_i - \tau_i^0(\ w_i)) > \mathrm{U}(1,\ w_i)$

which implies $\ 0 \leq \tau_i^1(\ w_i) <$ (U(0, w_i)-U(1, w_i) (4)

It is easy to check that if a condition is not fulfilled, everybody will always prefer y= 0 or y=1 and preference cannot be accessed.

One notes in particular that the transfer can never negative, which would otherwise push systematically the ex-post utility above or below the initial one for any stakeholder and lead to cheat.

So the declaration of sincere preference is made by stating a preference for 0 (respectively 1) as if there were no transfer, assorted with the maximum amount $x_i^*$of IWU that the stakeholder would pay for preserving the status-quo (respectively moving to the project.)

So the procedure should operate as follows not to violate strategy-proofness at this stage: (necessary but not sufficient condition)

The question posed to the stakeholder is: “*what is you preferred outcome and how much would you would be willing to pay at most to secure your preferred outcome.*

The reply x_i is interpreted as the declared measure of intensity of preference. Strategy-proofness requires that the sincere sign (- for 0 and or + for 1) be attributed to the sincere absolute value of this intensity $|x_i|$.

In short, each agent declares a real number x in]-∞ , +∞[and the procedure operate as follows:

Noting $\boldsymbol{x} = (x_1,\dots,x_i,\dots,x_n) \in \boldsymbol{R^n}$ the joint-message collected by a central planner, called a “profile”, one should have

x_i positive can be interpreted as *willingness to pay for the project* and $x_i$negative as *willingness to pay against it.*

One note also $\boldsymbol{x_{-i}}$ the projection of $\boldsymbol{x}$ on $\boldsymbol{R^{n-1}}$, i.e. the vector of all messages except that of i, also called “sub-profile” without i.

the decision rule is a function $\delta(\boldsymbol{x}) : \boldsymbol{R^n} \to \lfloor 0,1 \rfloor$ with:

$\delta(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \boldsymbol{R_0^n}$ a certain subset of $\boldsymbol{R^n}$

$\delta(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in \boldsymbol{R_1^n}$ the complement of $\boldsymbol{R_0^n}$ in $\boldsymbol{R^n}$

Note that because of the monotonicity of U these two subsets are dense in $\boldsymbol{R^n}$

And the transfers

$\boldsymbol{\tau}(\boldsymbol{\delta}(\boldsymbol{x})) : \boldsymbol{R^n} \to \boldsymbol{R^n}$ is the vector of rights to be paid, with $\tau_i(\boldsymbol{x})$ being the i^{th} component of $\boldsymbol{\tau}(\boldsymbol{x})$.

Subject to the necessary condition:

$y = 1$ and $x_i > 0 \Rightarrow x_i > \tau_i(\boldsymbol{x}) \geq 0$ (possibly paying, for satisfied with the project)
y $= 1$ and $x_i < 0 \Rightarrow \tau_i(\boldsymbol{x}) = 0$ (no payment if unsatisfied)
$y = 0$ and $x_i < 0 \Rightarrow x_i < -\tau_i(\boldsymbol{x}) \leq 0$ (possibly paying, for satisfied with the status-quo)
y $= 0$ and $x_i > 0 \Rightarrow \tau_i(\boldsymbol{x}) = 0$ (no payment if unsatisfied
$x_i = 0 \Rightarrow \tau_i(\boldsymbol{x}) = 0$ (no payment if indifferent)
$\delta(\boldsymbol{x}) : \boldsymbol{R^n} \to \lfloor 0,1 \rfloor$ provides the social outcome.
A first condition to preference revelation, i.e. to

Now, to explore new necessary conditions, we posit three additional properties for an acceptable procedure.

(a) Effectiveness of each stakeholder:

For each stakeholder there should be at least one profile $\boldsymbol{x}_{-i}$ such that there exist two messages x_i^1 and x_i^2 such that $\delta(\boldsymbol{x}_{-i}, x_i^1) \neq \delta(\boldsymbol{x}_{-i}, x_i^2)$. In other words, the agent can be decisive.

Of course, if for some stakeholder this condition is not satisfied, she has no incentive to disguise the truth, which remains therefore a trivial optimal strategy. However we want to specialize our study to "democratic" procedures in which each agent counts. Hence this condition, which in particular precludes dictatorship of a single stakeholder or of a subgroup. However, (a) does not imply that all agents are given the same weight in the determination of y, and it is easy to treat unequally different stakeholders by supposing a replication of certain. So, in the rest of this paper we set the condition:

(b)* Anonymity:*

For any two joint-messages $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ such that their components are in circular permutation, the social outcome and the transfers are unchanged.

(c) monotonicity

For any two profiles $\boldsymbol{x}^1$and $\boldsymbol{x}^2$ and such that $\boldsymbol{x}^1_{-i} \equiv \boldsymbol{x}^2_{-i}$

$$[\delta(\boldsymbol{x}^1) = 1 \text{ and } x_i^1 < x_i^2] \Rightarrow [\delta(\boldsymbol{x}^2) = 1]$$
$$[\delta(\boldsymbol{x}^1) = 0 \text{ and } x_i^1 < x_i^2] \Rightarrow [\delta(\boldsymbol{x}^2) = 0]$$

This condition, sometimes called "non-perverse counting", is analogous to the condition used in Arrow's theorem, and stated in many different forms by his followers.

Necessary condition for strategy-proofness

Since the procedure is anonymous, any circular permutation of the n messages received by the central planner should be treated alike, and therefore there is no need of indexing x . Any joint-message $\boldsymbol{x}$ can be split in two parts and written $(\bar{\boldsymbol{x}}.x)$ **w**here $\bar{\boldsymbol{x}}$ is the vector of remaining components of $\boldsymbol{x}$ (previously noted $\boldsymbol{x}_{-i}$) when component x is removed.

We may invoke the classical "revelation principle" to restrict the possible messages of the agent as their sincere preference.

A first (and well-known) property for x to be a dominant strategy i', is that outcome δ($(\bar{\boldsymbol{x}}.x)$ should depend only on $\bar{\boldsymbol{x}}$ and the sign of ($\bar{\boldsymbol{x}}$- x) and not on the value of x. In effect, for $\bar{\boldsymbol{x}}$ given, if one can find $x' \neq x$ such that the outcome remains the same (i.e. $\delta(\bar{\boldsymbol{x}}.x) = \delta(\bar{\boldsymbol{x}}.x')$) while the transfers are different ($\tau(\bar{\boldsymbol{x}}.x) \neq \tau(\bar{\boldsymbol{x}}.x')$, then either x or x' cannot be a dominant strategy when respectively taken as sincere preference.

Now, since $\boldsymbol{R}$ is unlimited, and thanks to the monotonicity condition, it is always possible to find a vector $\bar{\boldsymbol{x}}$ in $\boldsymbol{R}^{\boldsymbol{n-1}}$ and a number $\hat{x}$ *in* $\boldsymbol{R}$ such that:

$x > \hat{x} \Rightarrow \boldsymbol{\delta}(\bar{\boldsymbol{x}}.x) = 1$

$x < \hat{x} \Rightarrow \boldsymbol{\delta}(\bar{\boldsymbol{x}}.x) = 0$

In effect, it suffices to select x large enough or small enough respectively.

If x has an opposite sign to τ there will be no transfer unless a non-sincere strategy is played . The interesting case is when they have the same sign. In this case, the transfer may take only two values $\hat{x}$ or 0. For x dominant strategy one must have

$\hat{x} * x > 0$ and $x > 0$ (then $\delta = 1$), $\Rightarrow 0 \leq \tau < \hat{x}$ for x dominant strategy

$\hat{x} * x > 0$ and $x < 0$ (then $\delta = 0$) $\Rightarrow 0 > -\tau > \hat{x}$

In a Groves mechanism, the transfer is exactly equal to the aggregate value $\hat{x}$ of others' declaration. Here it is just said that it should be below. However since x can take all values between $\hat{x}$ *and* $+\infty$ if x is above $\hat{x}$ and all values between -∞ and $\hat{x}$ if below $\hat{x}$, the procedure can be represented by any *monotonous increasing function* $f(.)$ which transforms **R** in a subset **S** of **R** and is provided with an operation with the same properties as the addition to preserve the incentive properties.

$f: \boldsymbol{R} \rightarrow \boldsymbol{S}$ and there exits an operation Δ in $\boldsymbol{S}$, i.e. a mapping of **S.S** in **S,** which has the same properties as the addition, or otherwise said, which is an automorphims of the addition on **, i.e.**

$f(x)\Delta f(\hat{x}) > f(0) \Rightarrow y = 1$

$f(x)\Delta f(\hat{x}) \leq f(0) \Rightarrow y = 0$

and

$f(0) \leq f(\tau) \leq f(\hat{x})$ if x>0 and x. $\hat{x}$ >0

$f(0) > f(-\tau) > f(\hat{x})$ if x > and x. $\hat{x}$ >0

Otherwise said, (**S**, Δ) is must be a commutative "group", provided with monotonicity, which is automorphic to the commutative group, (**R,** +).

Remind that the requested properties are: associativity, commutativity, existence of a neutral element and existence of a symmetrical element for each element.

Since the procedure is anonymous and should not depend on the identity of the sender of message x, it is necessary to use the same aggregation procedure for all $\hat{x}$ and x that has been used for aggregating the components of $\overline{\boldsymbol{x}}$ to compute $\hat{x}$.

In conclusion, the necessary condition for strategy proofness is that it should operate on the pivot principle with a treatment of messages according to some commutative group (**S**,Δ) automorphic to (**R**,+).

The decision can be viewed also as the result of the maximization of the aggregate values with operation Δ of the x declared by the agents. Of course, the Pareto-optimality condition as proposed in the standard VCG-type model is not satisfied in the sense that the result does not maximize the sum of the $x_i's$ but, as it is always the case, the existence of surpluses or deficit cannot make this rule as compelling as any other like maximizing the aggregate value of the image of the $x_i's$, which is done in the proposed procedures

5- Examples of BG-mechanism with binary collective choice

Consider any numerical function $f: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$: such that

1) $f(0) = 0$
2) $x > y \;\; \Leftrightarrow f(x) > f(y)$
3) $f(x) = -f(-x) \quad \forall x$
4) $f(x)$ is continuous on **Z**

The inverse function f^{-1} does exit and is also continuous

Next define the operation:

$$x \oplus y \;= f^{-1}(f(x) + f(y))$$

Associativity, commutativity and existence of the same neutral element (0) are easily checked. f(.) establishes a group automorphims between ($\boldsymbol{R}$, +) and ($\boldsymbol{R}$, $\oplus$) and for this reason we call it “pseudo-addition” or “pseudo-summation”.

Since $\boldsymbol{R}$ is ordered, GB-Mechanisms can be constructed with this type of operation. We select for illustration the following one which has interesting limit conditions

$$f(x) = \qquad x^{\alpha} \qquad\qquad if\; x > 0$$

$$f(x) = -|x|^{\alpha} \qquad if\ x < 0$$

Where alpha is any positive real number

For the sake of computation, the formula of the operation is:

$$X \oplus^{\alpha} y = (\frac{x-y}{|x-y|}).\left(\left|\frac{x}{|x|}.|x|^{\alpha} + \frac{y}{|y|}.|y|^{\alpha}\right|\right)^{(\frac{1}{\alpha})}$$

This simply expresses that x and y are raised to the α power ignoring their sign. Then the sign are reinstated and addition is performed. Next, the absolute value is raised at the inverse power before attributing the sign the greatest absolute value (x or y) to the result .

Note that α could be taken with a different value depending on the sign of x. This would be a way to give more or less weight to the opponents or the partisans. However, we keep α constant thereafter.

An interesting property is that varying α, one obtains a range of cases various cases whose extreme are interesting to consider. Batteau (1982) has shown also that:

- when α tends to zero the mechanism tends to a mere binary voting with all transfers to zero, i.e. a sheer referendum with a yes-or-no vote
- the standard Groves mechanism is obtained of with $\alpha = 1$
- when α tends to infinity the mechanism tends to the Clarke pivoting mechanism, used in the second-price auction.

When nearing the lower limit of α, those mechanisms look become close to referenda (i.e. close to a one-man-one-vote situation), with a slightly different weight in monotonic relationship with the declared welfare change.

When nearing the upper limit of α those mechanisms become close to second-price auction: the stakeholder with the highest declaration in absolute value determines the choice and pays a penalty almost equal to the aggregate declaration of the rest of the stakeholders

This result establishes a continuity in strategy-proofness between the strict monetary assessment and the political vote. In some sort, algebra allows to reconcile political and economic approaches!

We leave to the reader the study of the following pseudo-addition

$$x \oplus^{*} y = x + y + xy$$

Which involves some sort of cross-effect in the aggregation and means that the weights of agents with similar valuations will be increased, by comparison to a mere addition

The previous theorem suggests to look at strategy-proof mechanism, less as systems for financing public projects or commons management than as just s mean to know the sincere preferences, which can be important for public management. In our case, the amount of private good collected, always positive but impossible to redistribute to any stakeholders, marks the defective allocative efficiency of these mechanisms.

7 Another example with the automorphism linking addition and multiplication

Let f(x) = k_u*($e^{(a*x)}$-1) if $x < 0$

f(x) = - k_l*($e^{(-b*x)}$-1) if $x > 0$

for some positive k_u, k_l, a, b

This form allows a wide variety of aggregation procedures that preserve the incentive properties

10- Conclusion

We have shown that the basic incentive property for sincere revelation in a G-mechanisms in the setting exposed requires to perform aggregation of the declarations received by a central planner according to operations which are automorphism of the addition. Using any other operation of this type provides the same incentive property and allows a wide variety of procedure eliciting sincere preference, apparently not mentioned before in the literature except in Batteau (1982). We have called them BG-mechanisms and we have provided some examples. We have mentioned that it is possible to find procedures as closed as possible of a mere binary referendum where the one-man-one vote applies without compensations. In the other direction, one can find mechanisms where only one agent is charged according to a second price auction mechanism.

The condition of Pareto-optimality taken as the maximization of the $x_{i's}$ has no more sense in the context of these mechanisms as the claim that the *average* grade of a student over several courses is a better aggregation than the median or any other operation on the set of notes. As long as the Individual Welfare Units (IWU) received by the central planner cannot be redistributed, the maximization of the aggregation of values with operation Δ is no more questionable than any other. After all, the whole literature on social choice refers to Pareto condition expressed on preorders without any concerns for transfers, or even interval orders.

One could also imagine a general uniform (or not uniform) periodical distribution of units of IWU and the possibility of an exchange market. This would affect the values of the sincere preference over time but the interest is to know, at the moment of the decision, what is the present willingness to pay.

Their tradability should therefore require that the central planner starts auctioning them among the stakeholders or at a broader scale, which would have the same counter-incentive effect as redistribution.

We have seen also that procedures do exist as close as one wants of a one-man-one-vote referendum with negligible penalties and rarity of pivots as the size of the society increases. This means that it could be possible to access the strength of preference even without penalty, each respondent knowing that exaggerating strongly his or her declaration to favor one outcome increases the risk of having to pay more than sincerely expected at the end.

These mechanisms apparently do not resist to collusion of stakeholders but we conjecture that they theoretically do (the core of the cooperative game do exist and is made of the dominants strategies only. However the proof is not easy to give.

The purpose of such mechanism is clearly no to fund projects or commons, but to acquire a map of their effect on the welfare of a population of stakeholders. This leaves the choice for anybody to be self-declared and cast a ballot with some x on it. Indifferent people would vote or not without changing anything to the upshot. In particular in environmental conflicts, it could be possible to determine to which extent economic interests are concerned or if it is a conflict of thoroughly non-economic moral values (i.e. not amenable to economic valuation through some NMV method.).

Over time, for people who would be frequently pivot and therefore have to pay, this could be measure of their power in decision.

These mechanisms are based on the principle pivot-payer; i.e. the decision-makers only are charged. The interest of this "commutative group" approach is for designing systems of penalties which are not expressed in terms of money, but simply reflect changes in welfare suffered or benefited by the stakeholders, and to charge penalties only to the pivot ones. We think that this can open a way to formalize better the system of compensations which is used in some countries to externalize the liability of pivot stakeholders.

NOTE: An Excel Spreadsheet can be provided on demand to study the effect of various procedures for binary choice by varying the size of the stakeholder group, the asymmetrical form of aggregation operation and the form of the distribution of preference

References

Apt K., Conitzer V., Guo M., Markakis E. (2008) Welfare Undominated Groves Mechanisms. In: Papadimitriou C., Zhang S. (*eds*) *Internet and Network Economics. WINE 2008. Lecture Notes in Computer Science* vol 5385. Springer, Berlin, Heidelberg.

Arrow K.J. (1951) Social Choice and Individual Values, Yale University Press.

Batteau P. (1982) Revelation of Preferences for Public Goods: Aggregation by Abelian Operations, Journal of Economic Theory 27, 331-352.

Batteau P., Blin J.M. (1979) elements for new insights in the Gibbard-Satterwaite theorem in *Aggregation and Revelation of Preferences.* Laffont, J.J. (*ed.*). Elsevier North-Holland

Batteau P., Blin J.M., Monjardet B. (1981), Stability of Aggregation Procedures, Ultrafilters, and Simple Games, *Econometrica* Vol. 49, No. 2 (Mar., 1981), pp. 527-534.

Green J., Laffont J.J. (1977) Characterization of satisfactory mechanisms for the revelation of preferences for public goods. *Econometrica*, 45:427–438.

Aggregation and Revelation of Preferences. (1977) Laffont, J.J. (*ed.*). Elsevier North-Holland

Clarke E.H. MultiPart (1971) Pricing of Public Goods*, Public Choice*, vol. 11, pp. 17-33

Groves T. (1973), Incentives in Teams, Econometrica, vol. 41, 1973, pp. 617-631

Groves, T., J. Ledyard (1987), Incentive Compatibility since 1972, in: Groves T. and Radner R. (Eds.), *Essays in Honor of Leonid Hurwicz,* University of Minnesota Press, Minneapolis,

Groves T., Loeb M. (1975), "Incentives and Public Inputs", *Journal of Public Economics,* vol. 4,. 211-226

Groves T, Ledyard J. (1977), Optimal Allocation of Public Goods: A Solution to the Free Rider Problem, *Econometrica,* vol. 45, 783-809.

Hurwicz L. (1972), "On Informationally Decentralized Systems", in McGuire M.C. et Radner R. (Eds.), Decision and Organization, North-Holland, Amsterdam, 1972, pp. 297-336

R. Porter, Y. Shoham, M. Tennenholtz. Fair imposition. *Journal of Economic Theory,*118:209–228, 2004. Early version appeared in IJCAI-01.

Rawls, J., (1971) *A Theory of Justice,* Harvard University Press, (1999 revised)

Vickrey W. (1961), Counterspeculation, Auctions, and Competitive Sealed Tenders, *Journal of*

Example of transfer for different values of α

The individual preferences for six individuals are 1, 2, 0,-10 -5, 7

On may observe the fast convergence toward a procedure type "second price auction" when α grow, while as α decreases, one moves towards an equal value for each participant and therefore an aggregation performed by sheer couting

	individual #	#1	#2	#3	#4	#5	#6	sum of values	pseudo sum	1	2	3	4	5	6
	sincere preferences	1,0	3,0	-	- 10,0	- 5,0	7,0			transfers	transfers	transfers	transfers	transfers	transfers
α=	0,1	-0,00	-0,00	/	/	/	-0,00	-4	0,34	-0,00	- 0,00	/	/	/	- 0,00
α=	0,2	-0,00	-0,03	/	/	/	-0,19	-4	0,25	-0,00	- 0,03	/	/	/	- 0,19
α=	0,3	-0,06	-0,52	/	/	/	-1,97	-4	0,15	-0,06	- 0,52	/	/	/	- 1,97
α=	0,4	-0,39	-1,70	/	/	/	-4,74	-4	0,06	-0,39	- 1,70	/	/	/	- 4,74
α=	0,5	/	/	-0,00	-9,87	-4,91	/	-4	-0,00	/	/	-0,00	-9,87	-4,91	/
G-Mech. α=	1	/	/	/	-6,00	-1,00	/	-4	-4,00	/	/	/	-6,00	-1,00	/
α=	2	/	/	/	-5,83	/	/	-4	-8,12	/	/	/	-5,83	/	/
α=	3	/	/	/	-6,27	/	/	-4	-9,10	/	/	/	-6,27	/	/
α=	4	/	/	/	-6,57	/	/	-4	-9,50	/	/	/	-6,57	/	/
α=	5	/	/	/	-6,74	/	/	-4	-9,70	/	/	/	-6,74	/	/
α=	6	/	/	/	-6,84	/	/	-4	-9,82	/	/	/	-6,84	/	/